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An algorithm for the local exhaustive search for alternatives in an essentially non-linear eigenvalue problem☆

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ABSTRACT

An algorithm for the local exhaustive search for alternatives, which enable one to reduce the number of auxiliary linear eigenvalue problems that must be solved is illustrated using the example of the problem of the supercritical behaviour of a longitudinally compressed rod at the interface of two dissimilar Winkler media. The basic principle of the algorithm consists of developing a qualitatively adequate natural form based on a complete exhaustive search for alternatives on a "widely spaced" grid with further subsequent doubling of the number of its nodes and exhaustively searching for alternatives only in the region of the roots of the natural form.

1. Introduction

The problem of the stability and supercritical behaviour of thin-walled structural components at the interface of dissimilar Winkler media reduces to investigating an operator equation of the form

$$\mathcal{A}u \equiv Au + C_1u_+ + C_2u_- = \lambda Qu \tag{1.1}$$

where A and Q are operators, which act from a certain Hilbert space, C_1 and C_2 are multiplication operators acting on the non-negative function and $u_+ = \max\{0, u\}$, $u_- = \min\{0, u\}$ are sections of the function u.

The operator A is positively homogeneous, i.e., such that $A(\alpha u) = \alpha Au$, if $\alpha>0$. Consequently, problems of the stability of structural components for unilateral couplings in the form of Winkler media reduce to the problem of the eigenvalues of a positively homogeneous operator. To solve eigenvalue problems with an operator equation of the form (1.1) a local method of searching for the eigenvalues of a positively homogeneous operator (henceforth called the local method) was proposed in Ref. 1. A proof of the convergence of the method was given as well as examples of its application.[2–4] The method is called a local method because it enables one to find any (local) minimum of the functional corresponding to Eq. (1.1). The convergence of the method was proved with extremely strict requirements imposed on the operators A and Q.[2] The search for a global minimum can be reduced to a problem of separable programming, to solve which one can use a theoretical local-method scheme in combination with the branch and boundary method.[5]

The algorithm for the complete exhaustive search for alternatives for solving a one-dimensional eigenvalue problem with an operator equation of the form (1.1) consists[2] of a finite-dimensional approximation of Eq. (1.1) and of finding, by exhaustive searching for alternatives, a non-contradictory natural form, to which the minimum eigenvalue corresponds. In general, this algorithm enables one to obtain not only the first eigenvalue, but also part of the discrete eigenspectrum (depending on the dimension of the grid). However, the practical use of this algorithm encounters difficulties related to the fact that when using it on a grid of dimension m it is necessary, when establishing the rule for finding alternatives, to solve 2^{m-1} linear eigenvalue problems.

In this paper we propose a combined algorithm for exhaustively searching for alternatives. Initially, on a narrow grid (i.e., such that 2^{m-1} is not a number that is too large) one realizes the complete exhaustive search algorithm and one chooses a qualitatively adequate natural form, i.e., one has a form of the graph that is stable as m increases (for example, the natural form with two half-waves). One then uses the local exhaustive search algorithm, which consists of the fact that the number of the grid nodes is successively doubled by dividing them in halves, and the exhaustive search is only carried out in the region of the roots of the natural form.

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Apart from those mentioned above, we are not aware of any other methods of solving an eigenvalue problem with the operator equation (1.1).

The local exhaustive search algorithm (more correctly, the combination of the complete exhaustive search algorithm and the local exhaustive search algorithm) is illustrated by solving the problem of the longitudinal stability of a rod at the interface of dissimilar Winkler media.

2. Formulation of the problem

We will derive an expression for the total potential energy of a "rod - Winkler media" system.

The bending strain energy of a rod is given by the formula

$$U_1 = \frac{1}{2}\int_V \sigma_{xx}\varepsilon_{xx}dV = \frac{EI}{2}\int_0^l \vartheta'^2 ds, \quad \vartheta' = \frac{d\vartheta}{ds}$$

The transverse reaction on displacement of an element ds of the rod is equal to $c_1 w ds$ if $w > 0$ and $-c_2 w ds$ if $w < 0$ (Fig. 1). In the first case the elementary strain energy of the medium is equal to $\frac{1}{2}c_1 w_+^2 ds$, and in the second case it is $-\frac{1}{2}c_2 w_-^2 ds$. The total strain energy of Winkler media is given by the formula

$$U_2 = \frac{1}{2}\int_0^l \left[c_1 w_+^2 + c_2 w_-^2\right] ds$$

Further, taking into account that the work of the compressive force P is given by the formula (see Fig. 1)

$$A_P = P\int_0^l (ds - dx) = P\int_0^l (1 - \cos\vartheta) ds$$

we obtain the following expression for the total potential energy of the system

$$\Pi(\vartheta, w) = U_1 + U_2 - A_P = \frac{1}{2}\int_0^l \left[EI\vartheta'^2 + c_1 w_+^2 + c_2 w_-^2 - 2P(1 - \cos\vartheta)\right] ds \tag{2.1}$$

Bearing in mind that, for small ϑ (see Fig. 1)

$$w' = \frac{dw}{ds} = \sin\vartheta \approx \vartheta, \quad dx = ds\cos\vartheta \approx ds, \quad w' = \frac{dw}{ds} \approx \frac{dw}{dx}$$

$$\cos\vartheta \approx 1 - \frac{1}{2}\vartheta^2 \approx 1 - \frac{1}{2}\left(\frac{dw}{dx}\right)^2, \quad 1 - \cos\vartheta \approx \frac{1}{2}\left(\frac{dw}{dx}\right)^2$$

expression (2.1) can be written in the form

$$\Pi(w) \approx \frac{1}{2}\int_0^l F(w, w', w'')dx; \quad F = EIw''^2 - Pw'^2 + c_1 w^2 \varkappa(w) + c_2 w^2 \varkappa(-w) \tag{2.2}$$

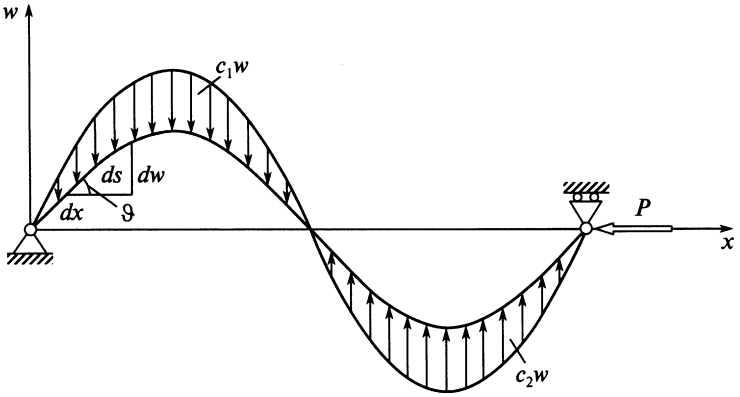

Fig. 1.

Here we have introduced the function

$$\kappa(w) = \begin{cases} 1, & w > 0 \\ 0, & w \le 0 \end{cases}$$

and a derivative with respect to x is denoted by a prime.

The Euler-Poisson equation for a functional of the type (2.2) has the form

$$\frac{\partial F}{\partial w} - \frac{d}{dx}\frac{\partial F}{\partial w'} + \frac{d^2}{dx^2}\frac{\partial F}{\partial w''} = 0$$

Taking into account the fact that

$$\frac{\partial F}{\partial w} = 2c_1 w\kappa(w) + 2c_2 w\kappa(-w) = 2c_1 w_+ + 2c_2 w_-$$

we arrive at the following Euler-Poisson equation for the integral (2.2), when EI = const:

$$EIw^{IV} + c_1 w_+ + c_2 w_- = -Pw''$$

$$c_1 = c_2 = c, \text{ то} \tag{2.3}$$

If $c_1 = c_2 = c$, we have

$$c_1 w_+ + c_2 w_- = c(w_+ + w_-) = cw$$

and Eq. (2.3) becomes the well-known equation for the longitudinal bending of a beam, resting on an elastic base (see, for example, Ref. 6, formula (3.106)).

Making the replacement of variables given by the formulae

$$\xi = \frac{\pi}{l}x, \quad \lambda = \frac{Pl^2}{\pi^2 EI}, \quad k_i = \frac{l^4}{\pi^4 EI}c_i, \quad i = 1, 2 \tag{2.4}$$

instead of the functional (2.2) we will consider the following functional

$$\Pi_1(w) \stackrel{\Delta}{=} \frac{l^3}{\pi^3 EI}\Pi(w) = \frac{1}{2}\int_0^\pi \left[w''^2 - \lambda w'^2 + k_1(w_+)^2 + k_2(w_-)^2 \right] d\xi, \quad w' \stackrel{\Delta}{=} \frac{dw}{d\xi} \tag{2.5}$$

We will replace formula (2.5) by an approximate formula using a discrete representation of the function $w(\xi)$, $\xi \in [0, \pi]$ on the grid (Fig. 2)

$$w_i \stackrel{\Delta}{=} w(\xi_i), i = 0, 1, 2, \ldots, m; \quad \xi_{i+1} = \xi_i + h, \quad h = \pi/(m+2), \xi_{-1} = 0, \xi_{m+1} = \pi$$

We will evaluate the integrals using the trapezoid quadrature formula, and we will approximate the derivatives by finite-difference relations

$$w'(\xi_i) = \frac{w_{i+1} - w_{i-1}}{2h}, \quad w''(\xi_i) = \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} \tag{2.6}$$

We will represent the values of the sections of the function $w(\xi)$ at the grid nodes by the formulae

$$w_+(\xi_i) = b_i w_i, \quad w_-(\xi_i) = (1 - b_i)w_i; \quad b_i = (w_i) \tag{2.7}$$

The boundary conditions of the hinged support

$$w(0) = w(\pi) = 0, \quad w''(0) = w''(\pi) = 0 \tag{2.8}$$

can be written, in terms of the discrete values of the function $w(\xi)$, in the form (see Fig. 2)

$$w_{-1} = w_{m+1} = 0, \quad w_0 = w_1/2, \quad w_m = w_{m-1}/2 \tag{2.9}$$

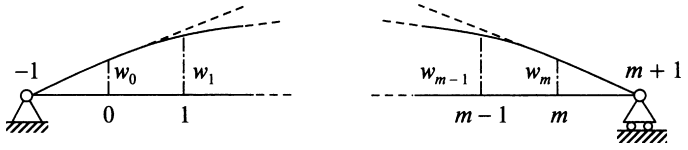

Fig. 2.

Taking relations (2.7) and (2.9) into account, the strain energy of the Winkler media can be represented by the formula (apart from a constant factor (see (2.5); the displacements w_{-1}, w_0, w_m, w_{m+1} are eliminated from consideration using Eqs (2.9))

$$\frac{h}{2}\left(\frac{5}{4}\left[k_1b_1^2+k_2(1-b_1)^2\right]w_1^2+\sum_{j=2}^{m-2}\left[k_1b_j^2+k_2(1-b_j)^2\right]w_j^2+\right.$$
$$\left.+\frac{5}{4}\left[k_1b_{m-1}^2+k_2(1-b_{m-1})^2\right]w_{m-1}^2\right)\stackrel{\Delta}{=}\frac{1}{2}\tilde{w}^TC\tilde{w} \tag{2.10}$$

where

$$\tilde{w}=[w_1,w_2,\ldots,w_{m-1}]^T$$

$$C=h\,\mathrm{diag}\left[1.25(k_1b_1^2+k_2(1-b_1)^2),k_1b_2^2+k_2(1-b_2)^2,\ldots\right.$$
$$\left.\ldots,k_1b_{m-2}^2+k_2(1-b_{m-2})^2,1.25(k_1b_{m-1}^2+k_2(1-b_{m-1})^2)\right]$$

and, by virtue of the definition of b_i, we can write b_i, $1-b_i$ instead of b_i^2, $(1-b_i)^2$.
Bearing formulae (2.6), (2.9) and (2.10) in mind, we obtain

$$\Pi_1(\tilde{w})=(\tilde{w}^TA\tilde{w}+\tilde{w}^TC\tilde{w}-\lambda\tilde{w}^TQ\tilde{w})/2 \tag{2.11}$$

where A and Q are the five-diagonal matrices

$$A=\frac{1}{h^3}\begin{Vmatrix}3.25&-3.5&1&&&&\\-3.5&6&-4&1&&&\\1&-4&6&-4&1&&\\&.&.&.&.&.&\\&&1&-4&6&-4&1\\&&&1&-4&6&-3.5\\&&&&1&-3.5&3.25\end{Vmatrix}$$

$$Q=\frac{1}{4h}\begin{Vmatrix}2.75&-0.5&-1&&&&\\-0.5&2&0&-1&&&\\-1&0&2&0&-1&&\\&.&.&.&.&.&\\&&-1&0&2&0&-1\\&&&-1&0&2&-0.5\\&&&&-1&-0.5&2.75\end{Vmatrix} \tag{2.12}$$

The necessary condition for a minimum of functional (2.11) has the form

$$\nabla_{\tilde{w}}\Pi_1=A\tilde{w}+C\tilde{w}-\lambda Q\tilde{w}=0 \tag{2.13}$$

3. The complete exhaustive search algorithm

To construct part of the eigenspectrum of Eq. (2.13) we can use the algorithm of complete exhaustive search (CES) of possible forms of bending corresponding to the grid assumed. The algorithm consists of the following:

1) all 2^{m-1} possible representations of the form vector

$$b=\left[b_1,\ldots,b_{m-1}\right]^T \tag{3.1}$$

are considered, taking into account the definition of b_i;
2) for each version of the form vector the eigenvalue problem of the deterministic equation (16) is solved (all the components of the form vector are known);
3) the eigenpair (the number and form) is stored, for which the form of the bending $\tilde{w}^{(i)}$ or $-\tilde{w}^{(i)}$ agrees with the chosen form vector.

Table 1

i	$\lambda^{(i)}$	$\tilde{w}^{(i)}$		
		$w_1^{(i)}$	$w_2^{(i)}$	$w_3^{(i)}$
1	64.063	0.440	−0.723	0.435
2	13.675	0.513	0.121–0.725	
3	17.012	0.548	0.602	0.458

Table 2

No	b			λ	$\tilde{w}$		
	b_1	b_2	b_3		w_1	w_2	w_3
1	0	0	0	18.304	−0.506	−0.600	−0.506
2	0	0	1	none	–	–	–
3	0	1	0	64.243	−0.442	0.715	−0.442
4	0	1	1	13.675	−0.513	0.121	0.725
5	1	0	0	none	–	–	–
6	1	0	1	64.390	−0.423	−0.743	0.423
7	1	1	0	13.675	−0.725	0.121	−0.513
8	1	1	1	16.408	−0.615	0.729	0.615

The minimum eigenvalue λ_1, to which the first critical force

$$P_{cr}^{(1)} \stackrel{\Delta}{=} P_{cr} = \lambda_1 \frac{\pi^2 EI}{l^2}$$

corresponds is of the greatest interest.

We will use the CES algorithm for the case when $m = 4$ (the number of alternatives is $2^{m-1} = 8$) when $k_1 = 16$ and $k_2 = 18$. Suppose, for example, the form vector has the form

$$b = [0,1,1]^T \tag{3.2}$$

Then Eq. (2.13) can be written as

$$\frac{1}{h^3}\begin{Vmatrix} 3.25 & -3.5 & 1 \\ -3.5 & 6 & -3.5 \\ 1 & -3.5 & 3.25 \end{Vmatrix}\begin{Vmatrix} w_1 \\ w_2 \\ w_3 \end{Vmatrix} + h\begin{Vmatrix} 22.5 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 20 \end{Vmatrix}\begin{Vmatrix} w_1 \\ w_2 \\ w_3 \end{Vmatrix} = \frac{\lambda}{4h}\begin{Vmatrix} 2.75 & -0.5 & -1 \\ -0.5 & 2 & -0.5 \\ -1 & -0.5 & 2.75 \end{Vmatrix}\begin{Vmatrix} w_1 \\ w_2 \\ w_3 \end{Vmatrix} \tag{3.3}$$

The eigenpairs of this equation are shown in Table 1.

Since when $\lambda = \lambda^{(i)}$, together with $\tilde{w}^{(i)} = [w_1^{(i)}, w_2^{(i)}, w_3^{(i)}]^T$, $-\tilde{w}^{(i)}$ is also a solution of Eq. (3.3), the eigenpair $(\lambda^{(2)}, -\tilde{w}^{(2)})$ corresponds to the form vector (3.2).

As a result of the CES, part of the eigenspectrum of Eq. (2.13) for $m = 4$ can be discerned from Table 2. It can be seen that the two form vectors

$$b' = [0,1,1]^T \text{ and } b'' = [1,1,0]^T$$

correspond to the minimum eigenvalue (in the assumed approximation) $\lambda_1 = 13.675$.

The following eigenvectors correspond to these forms

$$[-0.513, 0.121, 0.725]^T \text{ and } [0.725, 0.121, -0.513]^T \tag{3.4}$$

Clearly, by virtue of the symmetry of the structure about the point $\xi = \pi/2$, the eigenforms (3.4) are identical in their physical meaning.

The convergence of the CES algorithm for calculating the first eigenvalue is clear from Table 3; the bending forms are shown in Fig. 3 by the curves with corresponding numbers.

4. The local exhaustive search algorithm

It can be seen from Table 3 that when one changes from a grid with four nodes to a grid with eleven nodes the first eigenvalue is almost halved, whereas the qualitative configuration of the eigenform remains unchanged, retaining the graph in the form of a curve with two

Table 3

m	4	6	8	11
λ_1	13.675	10.024	9.193	8.755
Form of bending	1	2	3	4

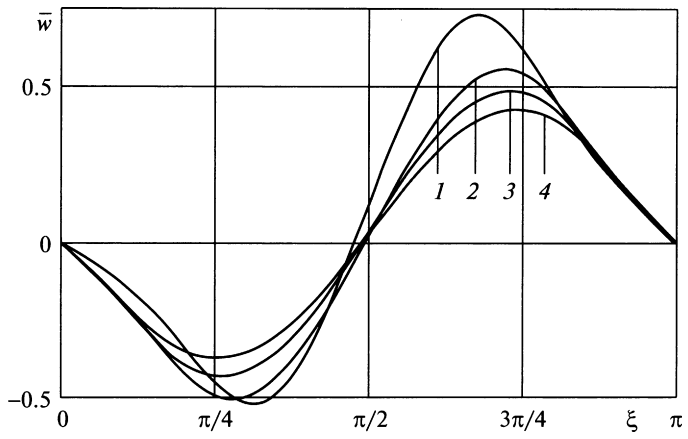

Fig. 3.

half-waves and a point of intersection of the ξ axis close to the middle of the section [0,π]. The eigenform having a stable form of the graph as *m* increases will be said to be qualitatively adequate.

Taking the above into account, we propose the following local exhaustive search (LES) algorithm:

1*l*) for a grid with a relatively small number of nodes (from 4 to 10) the CES algorithm is used and a qualitatively adequate eigenform is determined;

2*l*) the number of grid nodes is successively doubled by the dividing the intervals in half and a search is carried out for versions only close to the roots of the last approximation to the required eigenform; if the graph of the approximate eigenform does not intersect the ξ axis, a fine subdivision of the grid is carried out without exhaustive searching;

3*l*) the process is continued until the corresponding eigenvalue is stabilized with the required (and attainable) accuracy.

Two LES schemes can be employed.

The first scheme is based on the assumption that when the number of grid nodes is doubled the point of intersection by the graph of the approximate eigenform of the ξ axis does not go beyond the limits of the grid interval (see Fig. 2). In this scheme, for each root of the eigenvalue one obtains two versions of the calculations

$$1)\ b_{2i+1} = 1,\ 2)\ b_{2i+1} = 0$$

The remaining components of the form vector are not varied, i.e.

$$b_j = b_{j+1} = 1 \Rightarrow b_{2j} = b_{2j+1} = b_{2j+2} = 1,\quad b_k = b_{k+1} = 0 \Rightarrow b_{2k} = b_{2k+1} = b_{2k+2} = 0 \tag{4.1}$$

The second LES scheme is based on the assumption that when the number of grid nodes is doubled the point of intersection by the eigenform graph of the ξ axis can go beyond the limits of the interval in which it is situated up to double the number of grid nodes (Fig. 4). The second scheme reduces to searching for four versions for each root of the previous approximation to the corresponding eigenform:

$$1)\ b_{2i} = b_{2i+1} = b_{2i+2} = 0,\quad 2)\ b_{2i} = 1,\quad b_{2i+1} = b_{2i+2} = 0,$$

$$3)\ b_{2i} = b_{2i+1} = 1,\quad b_{2i+2} = 0,\quad 4)\ b_{2i} = b_{2i+1} = b_{2i+2} = 1$$

The remaining components of the form vector are not varied (see relation (4.1)).

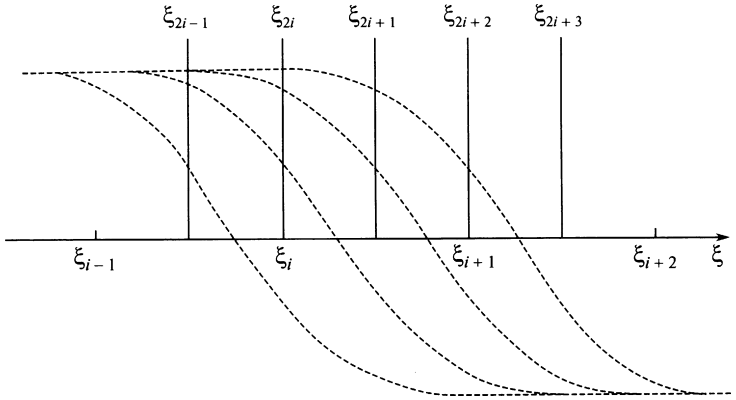

Fig. 4.

5. Numerical experiment

For the case $k_1 = 16$, $k_2 = 18$ considered in Section 2 we used the LES algorithm in the second scheme after realizing the CES scheme for $m = 11$. The corresponding results of the calculations are shown in Table 4.

Further, to calculate the first eigenvalue of Eq. (2.3) we used the first LES scheme, beginning with $m = 8$, for a fixed value of the parameter $k_2 = 20$ while varying the parameter k_1 from 10 to 20 (the case $k_1 = 16$, $k_2 = 18$ is an exception). The results of the calculations are shown in Table 5. The first eigenvalue for $m = 4$, to which the single half-wave eigenform corresponds, is indicated by an asterisk.

Remark. As calculations using the CES algorithm showed, for $m = 8$ and $m = 16$ the double half-wave form corresponds to the first eigenvalue, which is also taken as qualitatively accurate when $k_1 = 10$, $k_2 = 20$.

Comparing Tables 4 and 5, it can be seen that when $m = 256$ and $m = 352$ the first eigenvalue corresponding to the stiffness parameters $k_1 = 16$, $k_2 = 18$, has the same value of 8.231, up to the third decimal place, which can be taken as the final estimate of λ_1 for these values of the stiffness parameters. Here, when realizing the first local scheme (see Table 5) the combination of the CES algorithm ($m = 4, 8$) and the LES algorithm ($m = 16, 32, 64, 128, 256$) has been reduced to solving 146 linear eigenvalue problems instead of 2^{255} such problems, if the CES algorithm had been used. The use of the second local scheme (see Table 4) with a combination of the CES algorithm ($m = 11$) and the LES algorithm ($m = 22, 44, 88, 176, 352$) has been reduced to solving 1044 linear eigenvalue problems instead of 2^{351} such problems, if one had tried to use the CES method.

We will consider, in particular, the stability of a rod in a homogeneous elastic medium. The critical force and the form of bending corresponding to it are found from the well-known formulae (Ref. 6, formulae (3.15) and (3.16))

$$P_{\mathrm{cr}}^{(n)} = \pi^2 EIl^{-2}\left(n^2 + k/n^2\right), \quad w(\xi) = C\sin n\xi \tag{5.1}$$

The first formula of (5.1), taking notation (2.4) into account, leads to the relation

$$\lambda_{(n)} = n^2 + k/n^2 \tag{5.2}$$

where, by the second formula of (5.1), the subscript in parenthesis on the parameter λ indicates the number of half-waves of the natural form (note that, generally speaking, $\lambda_{(n)} \neq \lambda_n$, where $\lambda_{n-1} \le \lambda_n \le \lambda_{n+1}$.

From formula (5.2) with $k = 20$ we have

$$\lambda_{(1)} = 21, \quad \lambda_{(2)} = 9, \quad \lambda_{(3)} = 11.22, \quad \lambda_{(4)} = 17.25, \quad \lambda_{(5)} = 25.8$$

i.e., $\lambda_1 = \lambda_{(2)} = 9$. Using the finite-difference approximation with $m = 256$ we obtain a value for λ_1 equal to 9.007 (see Table 5). Hence we conclude that the values of λ_1 with $m = 256$ can be assumed to be reliable to within a few hundredths.

It follows from formula (5.2) that for any i, j a stiffness parameter k is obtained such that the condition $\lambda_{(i)} = \lambda_{(j)}$; in other words, the eigenvalue $\lambda_{(i)}$ for corresponding k will have a rank of 2. In fact, we have

$$i^2 + k/i^2 = j^2 + k/j^2$$

or

$$k = (ij)^2 \tag{5.3}$$

Two natural forms correspond to this multiple eigenvalue, namely, the i-half-wave and the j-half-wave forms.

It can be seen from formula (5.3) that if i and/or j are expanded in numerical factors, different eigenvalues correspond to the same k. For example, for $k = 36$ we have

$$\lambda_{(2)} = \lambda_{(3)} = 13, \quad \lambda_{(1)} = \lambda_{(6)} = 37$$

The minimum stiffness parameter, which has a second-rank eigenvalue, will be $k = 4$, for which $\lambda_1 = \lambda_{(2)} = 5$. If $k < 4$, the natural form with a single half-wave corresponds to the first eigenvalue while the natural form with two half-waves corresponds to $k > 4$.

Table 4

m	11	22	44	88	176	352
λ_1	8.755	8.370	8.271	8.239	8.234	8.231

Table 5

m	λ_1				
	$k_1 = 10\ k_2 = 20$	$k_1 = 14\ k_2 = 20$	$k_1 = 16\ k_2 = 18$	$k_1 = 19\ k_2 = 20$	$k_1 = k_2 = 20$
4	10.721*	13.228	13.675	14.448	14.603
8	8.069	9.048	9.193	9.890	10.024
16	7.466	8.392	8.563	9.217	9.349
32	7.232	8.140	8.309	8.956	9.086
64	7.184	8.082	8.249	8.892	9.023
128	7.171	8.066	8.235	8.874	9.009
256	7.165	8.062	8.231	8.871	9.007

Table 6

m	λ_1	λ_2	λ_3	Algorithm
8	4.996	9.917	14.033	CES
16	4.981	5.897	10.707	LES
32	4.961	5.042	9.675	LES
64	4.956	5.009	9.487	LES
128	4.950	5.000	9.452	LES
256	4.950	4.999	9.444	LES

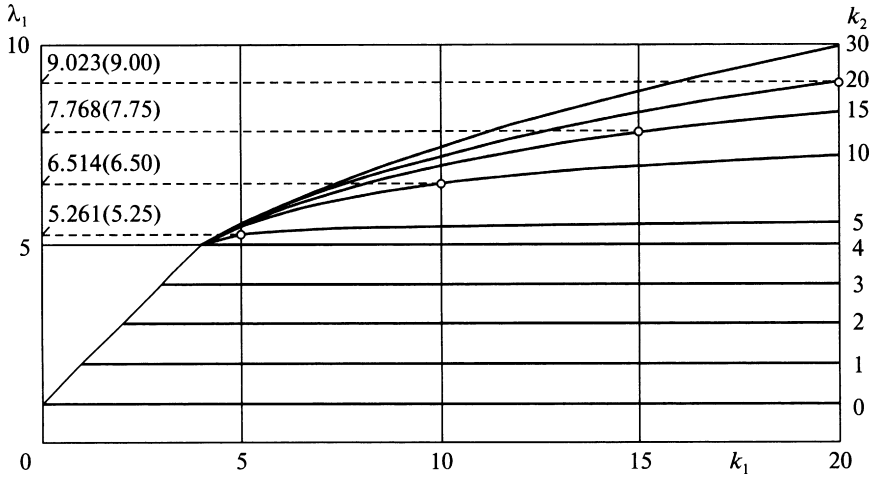

Fig. 5.

Taking this into account, we conclude that when $k_1 \to 4$, $k_2 = 4$, the eigenvalues will approach values defined by the formula (see (5.2)) $\lambda_{(n)} = n^2 + 4/n^2$: $\lambda_1 = \lambda_{(1)} = 5$, $\lambda_2 = \lambda_{(2)} = 5$, $\lambda_{(3)} = 9^{4/9}$ etc. In Table 6 we show the convergence of the combination of the CES algorithm and the LES algorithm (the first LES scheme was used) for the first three eigenvalues with $k_1 = 3.95$ and $k_2 = 4$. Since the eigenvalue in this case corresponds to the natural form with a single half-wave, in the same way in the case of a homogeneous medium, having a stiffness parameter $k = 3.95$, we conclude on the basis of Table 6 that λ_1 reaches an exact value for $m = 128$ (in the format used here for writing the eigenvalues). As regards the eigenvalues to which the natural forms in the form of two or three half-waves correspond, the following estimates hold for them

$$4.987 < \lambda_2 < 5.000, \quad 9.439 < \lambda_3 < 9.444$$

It should be noted that, to obtain the eigenvalues $\lambda_{(n)}$, $n > 3$ it is necessary to realize the CES algorithm on a denser grid, i.e., for $m > 8$. In other words, the CES algorithm, realised with $m = 8$ for $k_1 = 3.95$, $k_2 = 4$, only enables one to obtain the first three qualitatively adequate natural forms.

In Fig. 5 we show graphs of the function $\lambda_1(k_1)$, calculated using a combination of the CES and LES algorithms on a grid with $m = 64$ for fixed values of $k_2 (0 \le k_2 \le 30)$. The fact that the above graphs smoothly change into points, corresponding to $k_1 = k_2 \stackrel{\Delta}{=} k = 5, 10, 15, 20$ (see formula (5.2)), indirectly confirms the correctness of the calculations (the exact values of $\lambda_1(k)$ are shown in parenthesis in Fig. 5).

If $k_1 \le 4$ and/or $k_2 \le 4$, one obtains a natural form in the form of a single half-wave, curved when $k_1 \ne k_2$ towards the elastic medium of lesser stiffness. If, for example, $k_2 = 3$, then for any $k_1 > 3$ the first eigenvalue is given by the formula $\lambda_1 = \lambda_{(1)} = 1 + k_2 = 4$ (see Fig. 5). The rod in this case is bent towards a medium with stiffness k_2 (i.e., towards the medium with negative values of $w(\xi)$). If $k_1 \le k_2 = 3$, the rod is bent towards positive values of the function $w(\xi)$, and the first eigenvalue depends linearly on k_1, i.e., $\lambda_1 = 1 + k_1$, $0 \le k_2 \le 3$.

If the stiffness parameters k_1 and k_2 exceed a value of 4, but are not too large, one obtains a natural form in the form of two half-waves. In this case, for any fixed parameter k_2, the first eigenvalue is a continuous function of k_1 (see Fig. 5).

In Table 7 and Fig. 6 we show the results of calculations of the first eigenvalue and of the eigenform corresponding to it using the local method with $m = 34$ [3] and a combination of the CES algorithm ($m = 8$) and the LES algorithm ($m = 16, 32$). The parameter k_1 was fixed ($k_1 = 16$) while the parameter k_2 was increased from $k_2 = 18$ to $k_2 = 810$. In Fig. 6 it can be seen that the half-wave $w < 0$ as k_2 increases is, as it were, displaced towards the medium with lower stiffness. Taking into account the fact that each of the methods compared gives an upper estimate of the eigenvalue, we see from Table 7 that the combined method of exhaustive searching gives a more exact estimate for the first eigenvalue, since $\lambda_1(k_2) < P_l(k_2)$ (for the first eigenvalue, calculated by the local method, the notation P_l used previously in Ref. 3, is retained).

Table 7

k_2	18	90	150	810
P_l	8.338	9.842	9.956	10.105
λ_1	8.309	9.797	9.909	10.055
Form of bending	1	2	3	4

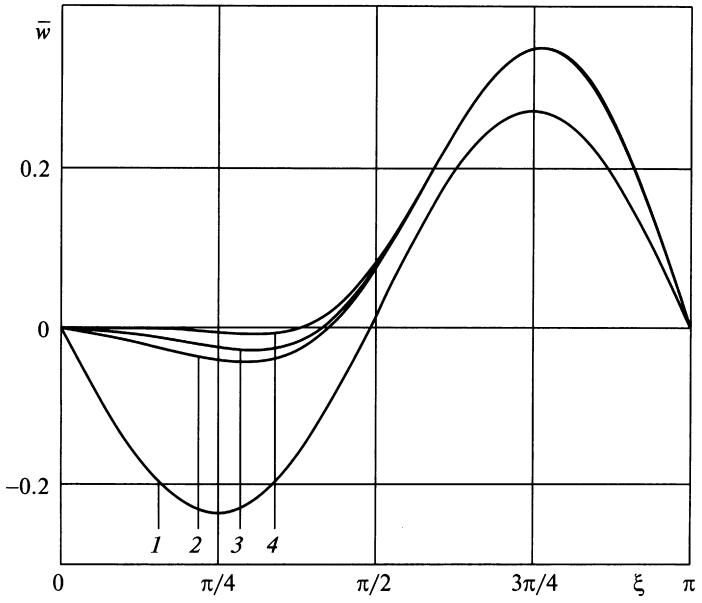

Fig. 6.

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